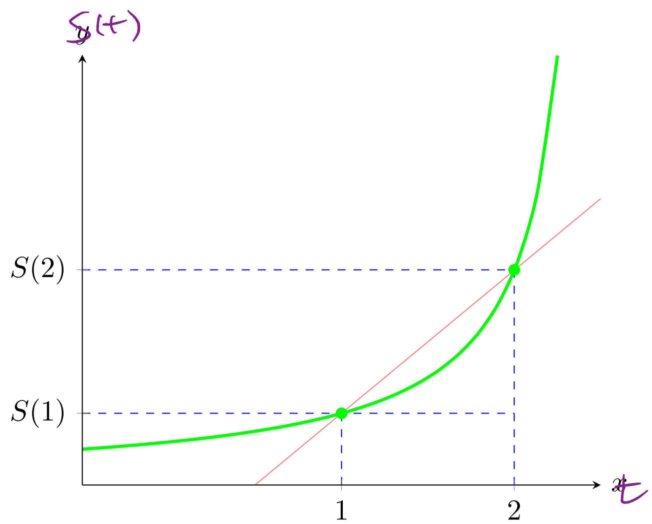
MATH1520 University Mathematics for Applications — Fall 2021

Chapter 4: Differentiation I

Learning Objectives:
(1) Define the derivatives, and study its basic properties.
(2) Study the relationship between differentiability and continuity.
(3) Use the constant multiple rule, sum rule, power rule, product rule, quotient rule and chain rule to find derivatives.
(4) Explore logarithmic differentiation.

4.1 Motivation & Definition

Motivation from physics: velocity Suppose an object is moving along x-axis from the origin to right. Let $S = S(t)$ be the position of the object at time t. What is the average velocity of this object from $t = 1$ to $t = 2$?



$$\begin{aligned}
\text{Average velocity from } t=1 \text{ to } t=2 &= \frac{\text{Change of position}}{\text{Change of time}} \\
&= \frac{\Delta S}{\Delta t} \\
&= \frac{S(2)-S(1)}{2-1} \\
&= \text{slope of secant line passing through } (1, S(1)) \text{ and } (2, S(2))
\end{aligned}$$

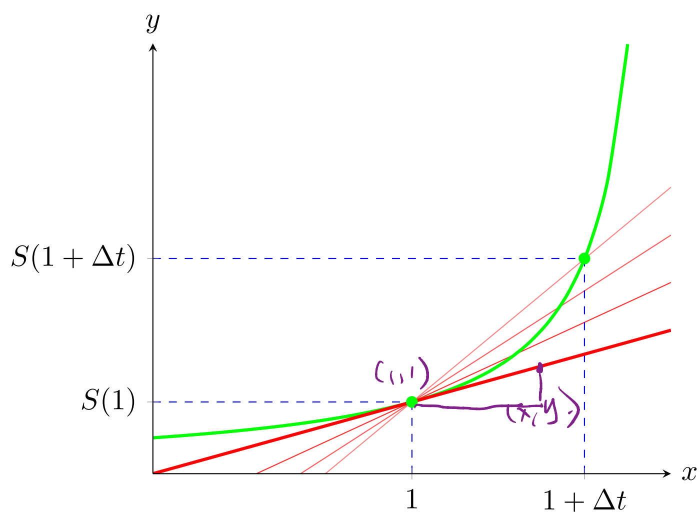
Question: What is the instantaneous velocity at $t = 1$?

Idea: Average velocity from $t = 1$ to $t = 1 + \Delta t$ is $\dfrac{S(1+\Delta t) - S(1)}{\Delta t}$, where Δt is small.

Let $\Delta t \to 0$, the instantaneous velocity at $t = 1$ is defined to be

$$S'(1) = \lim_{\Delta t \to 0} \frac{S(1+\Delta t) - S(1)}{\Delta t},$$

which is called the **derivative** of S at $t = 1$. $S'(1)$ describes the rate of change of $S(t)$ at $t = 1$.



Remark. Terminology: The term "velocity" takes the direction of motion into account; it can be positive or negative. The term "speed" only takes into account the rate of change, disregarding the direction. It is the absolute value of the velocity.

Definition 4.1.1. The **derivative** of $f(x)$ is the function

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}. \tag{4.1}$$

The process of computing the derivative is called **differentiation**, and we say that $f(x)$ is **differentiable** at $x = x_0$ if $f'(x_0)$ exists; that is, $\lim_{\Delta x \to 0} \dfrac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$ exists.

Remark. 1. By definition, if $f(x_0)$ is not well-defined, we cannot define $f'(x_0)$. So $f(x)$ must not be differentiable at $x = x_0$.

2. Another equivalent formula:

$$f'(x_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}.$$

$x = x_0 + \Delta x$

so when $\Delta x \to 0$

$x \to x_0$

3. $$\frac{\Delta f}{\Delta x} = \frac{f(x) - f(x_0)}{x - x_0}$$

is called **difference quotient.**

4. $f'(x_0)$ describes the rate of change of $f(x)$ at $x = x_0$.

5. When we say that we use **the first principle** to find derivatives, we mean that we use the definition (4.1) to find the derivative. However, later we will learn faster techniques to find derivatives.

Geometrical interpretation of differentiation: $f'(x_0)$ is the slope of tangent line to the curve of $f(x)$ at $x = x_0$.

Example 4.1.1. Let $f(x) = x^2$. Then (i) prove that $f(x)$ is differentiable at $x = 1$; (ii) find $f'(1)$ and the equation of the tangent line to the graph of f at $x = 1$.

Solution. (i) By the definition, at $x = 1$

$$f'(1) = \lim_{\Delta x \to 0} \frac{f(1 + \Delta x) - f(1)}{\Delta x} = \lim_{\Delta x \to 0} \frac{(1 + \Delta x)^2 - 1^2}{\Delta x}$$
$$= \lim_{\Delta x \to 0} (2 + \Delta x)$$
$$= 2,$$

$= 1 + 2\Delta x + (\Delta x)^2 - 1$

$= 2\Delta x + (\Delta x)^2$

$= \Delta x(2 + \Delta x)$

So, f is differentiable at 1, and $f'(1) = 2$.

(ii) The tangent line passes through $(1, f(1)) = (1, 1)$ with slope $f'(1) = 2$. So, the equation of the tangent line is

take (x, y) on the tangent line

$$\frac{y - f(1)}{x - (1)} = 2.$$

$\dfrac{y-1}{x-1} = 2$

$y - 1 = 2(x-1) = 2x - 2$

Thus

$$y = 2x - 1.$$

■

Definition 4.1.2. If $f(x) : A \to \mathbb{R}$ is differentiable at every point $x \in A$, then $f(x)$ is said to be a differentiable function in A, and the derivative function $f'(x) : A \to \mathbb{R}$ is well-defined.

Example 4.1.2. Let $f(x) = x^2$. Prove that $f(x)$ is differentiable on $\mathbb{R}$, and find $f'(x)$.

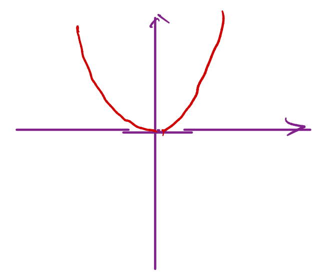
$a^2 - b^2 = (a+b)(a-b)$

Solution. For any $x \in \mathbb{R}$,

$(x+\Delta x+x)(x+\Delta x-x)$
$= (2x+\Delta x)\Delta x$

$$f'(x) = \lim_{\Delta x\to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x\to 0} \frac{(x+\Delta x)^2 - x^2}{\Delta x} = \lim_{\Delta x\to 0} (2x + \Delta x) = 2x.$$

So, f is differentiable at x, and $f'(x) = 2x$. ■



Notation: For $y = f(x) = x^2$,

$$f'(x) = \frac{dy}{dx} = \frac{df}{dx} = 2x; \quad f'(4) = \left.\frac{dy}{dx}\right|_{x=4} = \left.\frac{df}{dx}\right|_{x=4} = 2 \cdot 4 = 8.$$

$\parallel$

$\lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x}$ $\Delta \sim d.$

when the difference $\to 0$

Question Where does the minimum of x^2 occur? (Hint: what is the slope of the tangent line at the minimum?)

Example 4.1.3. Let $f(x) = \dfrac{x+1}{x-1}$. Using the definition of derivatives, compute $f'(x)$ for $x \neq 1$.

Solution.

$$f(x+\Delta x) - f(x) = \frac{x+\Delta x+1}{x+\Delta x-1} - \frac{x+1}{x-1}$$
$$= \frac{(x-1)(x+\Delta x+1) - (x+1)(x+\Delta x-1)}{(x-1)(x+\Delta x-1)}$$
$$= \frac{-2\Delta x}{(x-1)(x+\Delta x-1)}.$$

Therefore

$$f'(x) = \lim_{\Delta x\to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x\to 0} \frac{-2}{(x-1)(x+\Delta x-1)}$$
$$= \frac{\lim_{\Delta x\to 0}(-2)}{\lim_{\Delta x\to 0}(x-1)(x+\Delta x-1)} = \frac{-2}{(x-1)^2}.$$

■

$= (x-1) \lim_{\Delta x \to 0} (x+\Delta x-1)$

$= (x-1)(x+0-1) = (x-1)^2$

Example 4.1.4. Find the derivative of $f(x) = \sqrt{x}$ for $x > 0$.

use $a^2 - b^2 = (a+b)(a-b)$

$\Delta x = x + \Delta x - x = (\sqrt{x+\Delta x} + \sqrt{x})(\sqrt{x+\Delta x} - \sqrt{x})$

Solution.

$$
\begin{aligned}
f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} &= \lim_{\Delta x \to 0} \frac{\sqrt{x+\Delta x} - \sqrt{x}}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{(\sqrt{x+\Delta x} - \sqrt{x})(\sqrt{x+\Delta x} + \sqrt{x})}{\Delta x(\sqrt{x+\Delta x} + \sqrt{x})} \\
&= \lim_{\Delta x \to 0} \frac{1}{\sqrt{x+\Delta x} + \sqrt{x}} = \frac{\lim_{\Delta x \to 0} 1}{\lim_{\Delta x \to 0} \sqrt{x+\Delta x} + \lim_{\Delta x \to 0} \sqrt{x}} \\
&= \frac{1}{2\sqrt{x}}.
\end{aligned}
$$

So, $\left(x^{\frac{1}{2}}\right)' = \frac{1}{2}x^{-\frac{1}{2}}, x > 0.$ $\quad = \frac{1}{\sqrt{x} + \sqrt{x}}$ ■

Example 4.1.5. Find the derivative of $f(x) = \sqrt[3]{x}$.
Hint: $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$.

Solution. For any $x \neq 0$,

$$
\begin{aligned}
\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} &= \lim_{\Delta x \to 0} \frac{\sqrt[3]{x+\Delta x} - \sqrt[3]{x}}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{(\sqrt[3]{x+\Delta x} - \sqrt[3]{x})((\sqrt[3]{x+\Delta x})^2 + \sqrt[3]{x+\Delta x} \cdot \sqrt[3]{x} + (\sqrt[3]{x})^2)}{\Delta x((\sqrt[3]{x+\Delta x})^2 + \sqrt[3]{x+\Delta x} \cdot \sqrt[3]{x} + (\sqrt[3]{x})^2)} \\
&= \lim_{h \to 0} \frac{x + \Delta x - x}{\Delta x((\sqrt[3]{x+\Delta x})^2 + \sqrt[3]{x+\Delta x} \cdot \sqrt[3]{x} + (\sqrt[3]{x})^2)} \\
&= \lim_{\Delta x \to 0} \frac{1}{(\sqrt[3]{x+\Delta x})^2 + \sqrt[3]{x+\Delta x} \cdot \sqrt[3]{x} + (\sqrt[3]{x})^2} \\
&= \frac{1}{3(\sqrt[3]{x})^2} = \frac{1}{3}x^{-\frac{2}{3}}.
\end{aligned}
$$

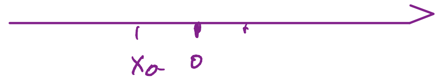
For $x = 0$,

$$
\lim_{\Delta x \to 0} \frac{f(0+\Delta x) - f(0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\sqrt[3]{\Delta x} - \sqrt[3]{0}}{\Delta x} = \lim_{\Delta x \to 0} \frac{1}{(\Delta x)^{\frac{2}{3}}} \quad \text{does not exist.}
$$

So,

$$
(x^{1/3})' = \begin{cases} \frac{1}{3}x^{-\frac{2}{3}}, & x \neq 0 \\ \text{Not exist at } x = 0, \text{ i.e. } x^{\frac{1}{3}} \text{ not differentiable at } 0 \end{cases}
$$

■



Example 4.1.6. Discuss the differentiability of $f(x) = |x|$. $= \begin{cases} x & \text{when } x \geq 0 \\ -x & \text{when } x < 0 \end{cases}$

Solution. For $x_0 > 0$,

$$f'(x_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{(x_0 + \Delta x) - x_0}{\Delta x} = 1.$$

For $x_0 < 0$,

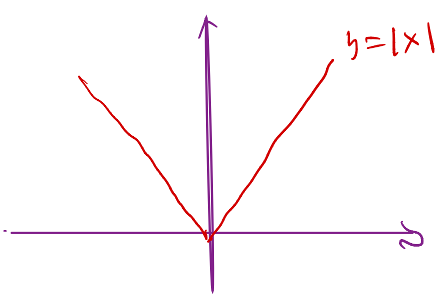
$$f'(x_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} == \lim_{\Delta x \to 0} \frac{-(x_0 + \Delta x) - (-x_0)}{\Delta x} = -1.$$

For $x_0 = 0$.

$$\lim_{\Delta x \to 0+} \frac{f(0 + \Delta x) - f(0)}{\Delta x} = \lim_{\Delta x \to 0+} \frac{\Delta x}{\Delta x} = 1.$$
$$\lim_{\Delta x \to 0-} \frac{f(0 + \Delta x) - f(0)}{\Delta x} = \lim_{\Delta x \to 0-} \frac{-\Delta x}{\Delta x} = -1.$$

so $\lim_{\Delta x \to 0} \frac{f(0+\Delta x) - f(0)}{\Delta x}$ does not exist.

$1 \neq -1$, so f is not differentiable at $x = 0$. So,



$$(|x|)' = \begin{cases} 1 & \text{if } x > 0, \\ \text{undefined} & \text{if } x = 0. \\ -1 & \text{if } x < 0, \end{cases}$$

■

4.2 Properties of derivatives

4.2.1 Differentiation and Continuity

Proposition 1. $f(x)$ is differentiable at $x = x_0 \implies f(x)$ is continuous at $x = x_0$.

$\lim_{x \to x_0} f(x) = f(x_0)$
$\Updownarrow$
$\lim_{x \to x_0} f(x) - f(x_0) = 0$
$\Updownarrow$
$\lim_{x \to x_0} (f(x) - f(x_0)) = 0$

Proof. Suppose $f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$ exists, then

$$\begin{aligned} \lim_{x \to x_0} (f(x) - f(x_0)) &= \lim_{x \to x_0} \left(\frac{f(x) - f(x_0)}{x - x_0} \cdot (x - x_0) \right) \\ &= \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \cdot \lim_{x \to x_0} (x - x_0) \\ &= f'(x_0) \cdot 0 = 0. \end{aligned}$$

So, $\lim_{x \to x_0} f(x) = \lim_{x \to x_0} (f(x) - f(x_0)) + \lim_{x \to x_0} f(x_0) = 0 + f(x_0) = f(x_0)$, that is, $f(x)$ is continuous at x_0. □

The converse is not true. For example, let $f(x) = |x|$. It is not differentiable at $x = 0$ but is continuous at $x = 0$.

Exercise 4.2.1. Let

$$f(x) = \begin{cases} x^2 - 1, & \text{if } x \geq 1 \\ 1 - x, & \text{if } x < 1 \end{cases}$$

(a) Show that $f(x)$ is continuous at $x = 1$.

(b) Show that $f(x)$ is differentiable everywhere except $x = 1$, and

$$f'(x) = \begin{cases} 2x, & \text{if } x > 1 \\ \text{undefined}, & \text{if } x = 1 \\ -1, & \text{if } x < 1 \end{cases}$$

continuous at $x = 1$

$$\lim_{x \to 1} f(x) = f(1) = 1^2 - 1 = 0 \checkmark$$

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (1 - x) = 0$$

$$\|$$

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (x^2 - 1) = 0$$

$$f'(1) = \lim_{\Delta x \to 0} \frac{f(1+\Delta x) - f(1)}{\Delta x} \quad \text{DNE}$$

$$\lim_{\Delta x \to 0^-} \frac{f(1+\Delta x) - f(1)}{\Delta x} = \lim_{\Delta x \to 0^-} \frac{1 - (1+\Delta x)}{\Delta x} = -1$$

$$\neq$$

$$\lim_{\Delta x \to 0^+} \frac{f(1+\Delta x) - f(1)}{\Delta x} = \lim_{\Delta x \to 0^+} \frac{f(1+\Delta x)}{\Delta x} = \lim_{\Delta x \to 0^+} \frac{(1+\Delta x)^2 - 1}{\Delta x} = \lim_{\Delta x \to 0^+} \frac{2\Delta x + \Delta x^2}{\Delta x} = \lim_{\Delta x \to 0^+} (2 + \Delta x) = 2$$

4.2.2 Differentiation and Arithmetic Operations

Theorem 2. *Let $f(x)$ and $g(x)$ be differentiable functions. Then*

(1) *Sum rule:* $(f+g)'(x) = f'(x) + g'(x)$.

(2) *Difference rule:* $(f-g)'(x) = f'(x) - g'(x)$.

(Leibniz rule) (3) *Product rule:* $\boxed{(fg)'(x) = f'(x)g(x) + f(x)g'(x).}$

(4) *Quotient rule:* $\boxed{\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}.}$ derived using the Leibniz rule and the chain rule

Proof. (1)

$$\begin{aligned}
(f+g)'(x) &= \lim_{\Delta x \to 0} \frac{(f+g)(x+\Delta x) - (f+g)(x))}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) + g(x+\Delta x) - (f(x) + g(x))}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} + \lim_{\Delta x \to 0} \frac{g(x+\Delta x) - g(x)}{\Delta x} \\
&= f'(x) + g'(x).
\end{aligned}$$

(3)

$$
\begin{aligned}
(fg)'(x) &= \lim_{\Delta x \to 0} \frac{f(x+\Delta x)g(x+\Delta x) - f(x)g(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{f(x+\Delta x)g(x+\Delta x) - f(x+\Delta x)g(x) + f(x+\Delta x)g(x) - f(x)g(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{f(x+\Delta x)g(x+\Delta x) - f(x+\Delta x)g(x)}{\Delta x} + \lim_{\Delta x \to 0} \frac{f(x+\Delta x)g(x) - f(x)g(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \left(f(x+\Delta x) \cdot \frac{g(x+\Delta x) - g(x)}{\Delta x} \right) + \lim_{\Delta x \to 0} \left(\frac{f(x+\Delta x) - f(x)}{\Delta x} \cdot g(x) \right) \\
&= \lim_{\Delta x \to 0} f(x+\Delta x) \cdot \lim_{\Delta x \to 0} \frac{g(x+\Delta x) - g(x)}{\Delta x} + \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \cdot \lim_{\Delta x \to 0} g(x) \\
&= f(x)g'(x) + f'(x)g(x).
\end{aligned}
$$

Remark. Here we used:

$$g(x) \text{ is differentiable at } x \quad \Rightarrow \quad g(x) \text{ is continuous at } x$$

so, $\lim_{\Delta x \to 0} f(x+\Delta x) = f(x)$.

□

Exercise 4.2.2. Prove other rules using the first principle.

Remark. 1. The product rule is more commonly referred to as the *Leibniz rule*.
Caveat: $(f \cdot g)' \neq f' \cdot g'$!
2. The quotient rule (4) can be derived from the Leibniz rule together with the chain rule (Section 4.3).

4.2.3 Derivatives of Elementary Functions

Theorem 3 (Constant functions).

$$\boxed{f(x) = k \quad \Rightarrow \quad f'(x) = 0}$$

Proof.

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{k-k}{\Delta x} = 0.$$

□

As a consequence, we have

$$\boxed{(kf(x))' = (k)'f(x) + kf'(x) = kf'(x),} \quad \text{for any constant } k.$$

Remark. It can also be proved by the first principle.

Theorem 4 (The Power Rule)**.**

$$\boxed{(x^a)' = ax^{a-1},} \quad \textit{whenever it is well-defined, } a \in \mathbb{R}.$$

Proof. We will only prove the special case when n is an integer.
Recall

$$x^n - y^n = (x-y)(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1}).$$

So

$$(x+\Delta x)^n - x^n = (x+\Delta x - x)((x+\Delta x)^{n-1} + (x+\Delta x)^{n-2}x + \cdots + (x+\Delta x)x^{n-2} + x^{n-1}).$$

We have

$$\begin{aligned}\lim_{\Delta x\to 0}\frac{(x+\Delta x)^n - x^n}{\Delta x} &= \lim_{\Delta x\to 0}((x+\Delta x)^{n-1} + (x+\Delta x)^{n-2}x + \cdots + (x+\Delta x)x^{n-2} + x^{n-1}) \\ &= x^{n-1} + x^{n-2}x + \cdots xx^{n-2} + x^{n-1} = nx^{n-1}.\end{aligned}$$

□

Remark. Alternatively, combine the fact that $x' = 1$ and the Leibniz rule.

Example 4.2.1.

$$\begin{array}{rcll}(x^3)' &=& 3x^2, & x \in \mathbb{R} \\ (\sqrt{x})' &=& \frac{1}{2}x^{-\frac{1}{2}}, & x > 0. \quad \text{Caution: } x \text{ can not be } 0. \\ (\sqrt[3]{x})' &=& \frac{1}{3}x^{-\frac{2}{3}}, & x \neq 0. \quad \text{Caution: } x \text{ can be negative.} \\ (x^{\frac{3}{2}})' &=& \frac{3}{2}x^{\frac{1}{2}}, & x > 0.\end{array}$$

Theorem 5 (Exponential functions and Logarithmic functions)**.**

$$\boxed{(e^x)' = e^x; \quad (a^x)' = a^x \ln a,} \qquad a > 0, a \neq 1, x \in \mathbb{R}.$$

$$\boxed{(\ln x)' = \frac{1}{x}; \quad (\log_a x)' = \frac{1}{x \ln a},} \qquad a > 0, a \neq 1, x > 0.$$

general base a case follows from base e
case via the chain rule

Proof. (Optional!)

$$(\ln x)' = \frac{1}{x} \iff \lim_{\Delta x \to 0} \frac{\ln(x+\Delta x) - \ln x}{\Delta x} = \frac{1}{x}$$

$$\iff \lim_{\Delta x \to 0} \frac{\ln(1+\frac{\Delta x}{x})}{\frac{\Delta x}{x}} = 1$$

$$\iff \lim_{y \to 0} \ln(1+y)^{\frac{1}{y}} = 1, \quad \text{(change variable: } y := \tfrac{\Delta x}{x}\text{)}$$

$$\iff \lim_{y \to 0} (1+y)^{\frac{1}{y}} = e$$

$$\iff \lim_{z \to +\infty} \left(1+\frac{1}{z}\right)^{z} = \lim_{y \to 0+} (1+y)^{\frac{1}{y}} = e \quad \text{(change variable:} z = \frac{1}{y}\text{)}$$

$$\text{and} \quad \lim_{z \to -\infty} (1+\tfrac{1}{z})^{z} = \lim_{y \to 0-} (1+y)^{\frac{1}{y}} = e \quad \text{(definition of } e\text{).}$$

e^x, $\ln x$ are "inverse functions"

i.e $e^{\ln x} = x$

$\ln e^x = x$

using this fact and the chain rule

$(e^x)' = e^x \Leftrightarrow (\ln x)' = \frac{1}{x}$

$$(e^x)' = e^x \iff \lim_{\Delta x \to 0} \frac{e^{x+\Delta x} - e^x}{\Delta x} = e^x$$

$e^{x+\Delta x} - e^x = e^x(e^{\Delta x} - 1)$

$$\iff \lim_{\Delta x \to 0} \frac{e^{\Delta x} - 1}{\Delta x} = 1$$

$$\iff \lim_{y \to 0} \frac{y}{\ln(1+y)} = 1, \quad (\text{ let } y = e^{\Delta x} - 1)$$

$\Leftrightarrow 1+y = e^{\Delta x} \Leftrightarrow \ln(1+y) = \Delta x$

$$\iff \lim_{y \to 0} \frac{\ln(1+y)}{y} = \frac{d \ln x}{dx}\bigg|_{x=1} = 1.$$

For general a: The formulae can be deduced from the preceding special case of $a = e$ using the chain rule (Section 4.3). □

Remark. 1. Instead of the definition given in Section 2.5, some books use $\lim_{y \to 0} (1+y)^{\frac{1}{y}}$ as the definition of e.

2. The formula for $(e^x)'$ and the formula for $(\ln x)'$ imply each other, as e^x and $\ln x$ are "inverse functions" of each other. (Cf. Chapter 5.)

Example 4.2.2.

1. $(\sqrt{x} + 2^x - 3\log_2 x)' = (\sqrt{x})' + (2^x)' - 3(\log_2 x)' = \frac{1}{2}x^{-\frac{1}{2}} + 2^x \ln 2 - \frac{3}{x \ln 2}$

$(x^{\frac{1}{2}})'$

2. $\dfrac{d}{dx}(x^2e^x) = \dfrac{d}{dx}(x^2)\cdot e^x + x^2\cdot\dfrac{d}{dx}(e^x) = (2x+x^2)e^x$

3. $\left(\dfrac{\sqrt{x}}{3^x}\right)' = ?$

   by the quotient rule: $\dfrac{(\sqrt{x})'3^x - \sqrt{x}(3^x)'}{(3^x)^2} = \dfrac{\frac12 x^{-\frac12}\cdot 3^x - x^{\frac12}\cdot 3^x\ln 3}{(3^x)^2} = \dfrac{\frac12 x^{-\frac12} - x^{\frac12}\ln 3}{3^x}$

   or, by Leibniz's rule: $\left(\sqrt{x}\cdot\left(\dfrac13\right)^x\right)' = \dfrac12 x^{-\frac12}\left(\dfrac13\right)^x + x^{\frac12}\left(\dfrac13\right)^x\ln\dfrac13 = \dfrac{\frac12 x^{-\frac12} - x^{\frac12}\ln 3}{3^x}.$

   $\dfrac{\sqrt{x}}{3^x} = x^{\frac12}\cdot\left(\frac13\right)^x$

   $\shortparallel$

   $(x^{\frac12})'\left(\frac13\right)^x + (x^{\frac12})\left(\left(\frac13\right)^x\right)'$

   $\ln\frac1a = -\ln a$

Exercise 4.2.3. Use two different methods to compute $\left(\dfrac{1-x^2}{\sqrt{x}}\right)'$.

Example 4.2.3. Suppose $f(x)$ and $g(x)$ are differentiable. Given $f(1)=1$, $f'(1)=2$, $g(1)=3$, $g'(1)=4$. Find the value of

$$\frac{d}{dx}\left(f(x)g(x)\right)$$

at $x=1$.

Solution. By the product rule

$$\frac{d}{dx}\left(f(x)g(x)\right) = f'(x)g(x)+f(x)g'(x).$$

At $x=1$, the above is

$$f'(1)g(1)+f(1)g'(1) = 2\times 3 + 1\times 4 = 10.$$

■

Example 4.2.4. Suppose $f(x)$, $g(x)$, $h(x)$ are differentiable. Compute

$$\frac{d}{dx}\left(f(x)g(x)h(x)\right).$$

Solution.

$$\begin{aligned}\frac{d}{dx}\left(f(x)g(x)h(x)\right) &= \left(f(x)g(x)\right)\frac{d}{dx}h(x) + h(x)\frac{d}{dx}\left(f(x)g(x)\right)\\ &= f(x)g(x)h'(x) + h(x)(f(x)\frac{d}{dx}g(x) + g(x)\frac{d}{dx}f(x))\\ &= f(x)g(x)h'(x) + f(x)g'(x)h(x) + f'(x)g(x)h(x).\end{aligned}$$

■

$= \dfrac{d}{dx}\left(f(x)\left(g(x)\,h(x)\right)\right) = f(gh)' + f'(gh)$

$= f(gh'+g'h) + f'(gh)$

4.3 The Chain Rule (for composite functions)

Theorem 6 (The Chain Rule)**.**

If $y = f(u)$ *is a differentiable function of* u*,*

$u = g(x)$ *is a differentiable function of* x*,*

then the composite function $y = f(g(x))$ *is a differentiable function of* x*, and*

$$\boxed{\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}}$$

or equivalently

$$\boxed{\frac{dy}{dx} = f'(g(x))g'(x).}$$

A heuristic explanation: Rewrite the difference quotient as a product: $\dfrac{\Delta y}{\Delta x} = \dfrac{\Delta y}{\Delta u} \cdot \dfrac{\Delta u}{\Delta x}$, then take $\Delta x \to 0$. (The notation "dx" is conventionally used to represent an "infinitesimal Δx".)

Example 4.3.1. Find

$$\frac{d}{dx}(1+2x)^5.$$

$y = (1+2x)^5 = u^5$

$u = 1+2x$

Solution. Set $y = f(u) = u^5$ and $u = g(x) = 1 + 2x$. Then $f(g(x)) = (1 + 2x)^5$.
By the chain rule,

$$f'(u) = \frac{dy}{du} = 5u^4 \qquad \text{and} \qquad g'(x) = \frac{du}{dx} = 2.$$

Hence,

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = (5u^4)(2) = 10(1+2x)^4,$$

or, alternatively written:

$$\frac{dy}{dx} = f'(g(x))g'(x) = 10(1+2x)^4.$$

■

$y = \sqrt{1+\sqrt{x}}$ let $u = 1+\sqrt{x}$
$= \sqrt{u}$

Example 4.3.2. Find

$$\frac{d}{dx}\sqrt{1+\sqrt{x}}.$$

Solution. Let $y = f(u) = \sqrt{u}$, $u = g(x) = 1+\sqrt{x}$. Then $f(g(x)) = \sqrt{1+\sqrt{x}}$.

$$\frac{dy}{du} = \frac{1}{2}u^{-1/2} = \frac{1}{2\sqrt{u}} \qquad \text{and} \qquad \frac{du}{dx} = \frac{1}{2\sqrt{x}}.$$

Therefore

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{1}{2\sqrt{u}}\frac{1}{2\sqrt{x}} = \frac{1}{4\sqrt{x}\sqrt{1+\sqrt{x}}}.$$

■

$(e^x)^y = e^{xy}$

Example 4.3.3. Using $(e^x)' = e^x$ and the chain rule, one may prove that $(a^x)' = a^x \ln a \quad (a > 0)$.

$a = e^{\ln a}$

Proof. Note that

$$\boxed{a^x = e^{\ln a^x}}$$ (Very useful technique!)

$y = a^x = (e^{\ln a})^x$
$= e^{x \ln a}$
$= e^u$
where $u = x \ln a$

Then,

$$\begin{aligned}(a^x)' &= (e^{\ln a^x})' \\ &= (e^{x\ln a})' \\ &= e^{x\ln a}\cdot \ln a \\ &= a^x \cdot \ln a.\end{aligned}$$

$\frac{da^x}{dx} = \frac{dy}{du}\frac{du}{dx}$
$= e^u \ln a.$

□

Exercise: Derive $(\log_a x)' = \frac{1}{x}\frac{1}{\ln a}$ from $(\ln x)' = \frac{1}{x}$ using the chain rule.

Example 4.3.4. Use the Leibniz rule and the chain rule to prove the quotient rule.

Proof. By the Leibniz rule, we have

$$\left(\frac{f}{g}\right)' = \left(f\cdot\frac{1}{g}\right)' = f'\cdot\frac{1}{g} + f\cdot\left(\frac{1}{g}\right)'.$$

For $\left(\frac{1}{g}\right)'$, let $y=\frac{1}{u}$, where $u=g(x)$. Then, by the chain rule,

$$\left(\frac{1}{g}\right)' = \frac{dy}{du}\cdot\frac{du}{dx} = -\frac{1}{g^2(x)}g'(x).$$

$\uparrow$ $-\frac{1}{g^2}\cdot g'$

Therefore,

$$\left(\frac{f}{g}\right)' = f'\frac{1}{g} - f\frac{g'}{g^2} = \frac{f'g-fg'}{g^2}.$$

□

Example 4.3.5. Find

$$\frac{d}{dx}e^{\sqrt{x^2+x}}.$$

$y = e^{\sqrt{x^2+x}} = e^u$

where $u = \sqrt{x^2+x}$

$= \sqrt{v}$

where $v = x^2+x$

Solution.

$$\frac{dy}{dx} = e^{\sqrt{x^2+x}}\cdot(\sqrt{x^2+x})' \qquad \text{(using the chain rule; write } y=e^u, u=\sqrt{x^2+x}\text{)}$$

$$= e^{\sqrt{x^2+x}}\cdot\frac{1}{2}(x^2+x)^{-\frac{1}{2}}\cdot(2x+1) \qquad \text{(using the chain rule again: let } u=\sqrt{w}, w=x^2+x\text{)}$$

$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = e^u\frac{du}{dx}$

■

$= e^u\frac{du}{dv}\frac{dv}{dx}$

$= e^u\,\frac{1}{2}\frac{1}{\sqrt{v}}(2x+1)$

Exercise 4.3.1. Prove that

1. $$\frac{d}{dx}(g(x))^n = n(g(x))^{n-1}g'(x).$$

2. $$\frac{d}{dx}e^{\sqrt{\frac{x-1}{x+1}}} = e^{\sqrt{\frac{x-1}{x+1}}}\cdot(x-1)^{-\frac{1}{2}}\cdot(x+1)^{-\frac{3}{2}}.$$

$y = e^{\sqrt{\frac{x-1}{x+1}}} = e^u$ where $u = \sqrt{\frac{x-1}{x+1}} = \sqrt{v}$ $v = \frac{x-1}{x+1}$

$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = e^u\frac{du}{dx} = e^u\frac{du}{dv}\frac{dv}{dx} = e^u\frac{1}{2\sqrt{v}}\frac{dv}{dx}$

$= e^u\frac{1}{2\sqrt{v}}\frac{1\cdot(x+1)-1\cdot(x-1)}{(x+1)^2} = \dots$

plug in u, v in x